FROM SIMPLICIAL LIE ALGEBRAS AND HYPERCROSSED COMPLEXES TO DIFFERENTIAL GRADED LIE ALGEBRAS VIA 1-JETS

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ABSTRACT. Let $\mathfrak{g}$ be a simplicial Lie algebra with Moore complex $N\mathfrak{g}$ of length k. Let G be the simplicial Lie group integrating $\mathfrak{g}$, such that each G_n is simply connected. We use the 1-jet of the classifying space $\overline{W}G$ to construct, starting from $\mathfrak{g}$, a Lie k-algebra L. The so constructed Lie k-algebra L is actually a differential graded Lie algebra. The differential and the brackets are explicitly described in terms (of a part) of the corresponding k-hypercrossed complex structure of $N\mathfrak{g}$. The result can be seen as a geometric interpretation of Quillen's (purely algebraic) construction of the adjunction between simplicial Lie algebras and dg-Lie algebras.

1. INTRODUCTION

In this paper we describe a geometric construction leading to Quillen's relation between simplicial Lie algebras and differential graded Lie algebras (DGLAs) [18]. We do that following the ideas of Ševera [21], which lead to a construction of L_∞-algebras (or, more generally, L_∞-algebroids) as 1-jets (differentiation) of simplicial manifolds. Here, we apply Ševera's construction to the case when the simplicial manifold in question is the classifying space $\overline{W}G$ of a simplicial Lie group G, the simplicial Lie group G having Moore complex of length k. Main results are the Proposition 5.2 and the Theorem 5.3. In Proposition 5.2 we describe explicitly the dg manifold representing the 1-jet functor $F_1^{\overline{W}G}$ and in Theorem 5.3 we describe explicitly the corresponding L_∞-algebra as a k-term differential graded Lie algebra L with the differential and brackets given in terms the hypercrossed complex structure of $N\mathfrak{g}$. The result is the same as the one described by the N-functor in the Quillen's adjunction between simplicial Lie algebras and dg-Lie algebras (see Proposition 4.4 of [18]. The construction can equivalently be viewed as an assignment of a k-term DGLA to a k-hypercrossed complex $\underline{\mathfrak{g}}$. The paper is organized as follows.

In Section 2, we recall the relevant material concerning simplicial Lie groups. In particular, we describe the Moore complex of a simplicial Lie group and illustrate its Lie hypercrossed complex structure in the low dimensional case of Lie crossed modules an Lie 2-crossed modules.

In Section 3, we recall the relevant facts regarding simplicial principal bundles.

In Section 4, we summarize Ševera's construction and give the relevant examples following [21]. In particular, we describe in detail the construction of a Lie algebra $\mathfrak{g}$ as a 1-jet of the classifying space BG of the corresponding Lie group G. Also, we describe in detail the construction of a Lie 2-algebra corresponding to a crossed module of Lie algebras $\mathfrak{h} \to \mathfrak{d}$ as a 1-jet of the functor associating to a surjective submersion $M \to N$ of (super)manifolds the set of $(H \to D)$-descent data over $M \to N$.

The examples mentioned above are the starting point of this paper. For this, we note that also the second example can be reinterpreted as the 1-jet of a simplicial manifold. The relevant simplicial manifold is the Duskin nerve of the strict Lie 2-group defined by the Lie crossed module $H \to D$, which is isomorphic to $\overline{W}G$, the classifying space of the simplicial Lie group associated to the Lie crossed module $H \to D$. Therefore, it is natural to generalize the above examples by applying Ševera's construction to the case of any simplicial Lie group G and describe explicitly the corresponding 1-jet of $\overline{W}G$. This is done in Section 5. The resulting dg manifold is described in Proposition 5.2 and the corresponding DGLA in Proposition 5.3. This DGLA is the same as the one described by Quillen in Sec. 4 of [18].

In this paper we do not discuss, up to occasional remarks,[1] applications to the higher gauge theory. These will be given in a forthcoming paper.

All commutators are implicitly assumed to be graded. Although we do not mention it explicitly, all constructions extend more or less straightforwardly to the case when all in involved Lie groups and Lie algebras are super. Hopefully, this is a wormless paper [15].

2. SIMPLICIAL GROUPS AND HIGHER CROSSED MODULES

Here we briefly sketch the relation between simplicial groups and hypercrossed complexes of groups. The basic idea comes from [7] and is further developed and formalized in [6]. We follow [7, 17, 11].

Although the above mentioned references[2] work with simplicial sets, the constructions and statements relevant relevant for our purposes can be straightforwardly formulated in the context of simplicial manifolds. Let G be a simplicial Lie group. We denote the corresponding face and degeneracy mappings ∂_i and s_i, respectively.

Definition 2.1. The *Moore complex* NG of G is the Lie group chain complex (NG, δ) with

$$NG_n := \bigcap_{i=1}^{n} \ker \partial_i$$

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[1]Cf. Remarks 5.6 and 5.7.

[2]and also our basic reference regarding simplicial objects [16] as well as other useful references [8, 13]

and the differentials $\delta_n : NG_n \to NG_{n-1}$ induced from the respective 0th face maps ∂_0 by restriction. It is a normal complex, i.e. $\delta_n NG_n$ is a normal subgroup of NG_{n-1}[3]. Of course, $NG_0 = G_0$. The Moore complex NG is said to be of length k if NG_n is trivial for $n > k$.[4]

The Moore complex NG carries a structure of a Lie *hypercrossed complex* structure, form which it can be reconstructed [7, 6]. To describe the idea behind this, we will need following lemma.

Lemma 2.2. *Let G be a simplicial Lie group. Then G_n can be decomposed as a semidirect product of Lie groups*

$$G_n \cong \ker \partial_n \rtimes s_{n-1} G_{n-1}.$$

Explicitly, for $g \in G_n$, the isomorphism is given by

$$g \mapsto (g s_{n-1} \partial_n g^{-1}, s_{n-1} \partial_n g).$$

The following proposition [7] is the result of a repetitive application on the above lemma.

Proposition 2.3. *For a simplicial Lie group G,*

$$G_n \cong (\ldots (NG_n \rtimes s_0 NG_{n-1})) \rtimes \cdots \rtimes s_{n-1} \ldots s_0 NG_1)$$

The bracketing an ordering of the terms should be clear from the first few terms of the sequence:

$$G_1 \cong NG_1^0 \rtimes s_0 NG_0^0$$
$$G_2 \cong (NG_2^0 \rtimes s_0 NG_1^0) \rtimes (s_1 NG_1^0 \rtimes s_1 s_0 NG_0^0)$$
$$G_3 \cong ((NG_3^0 \rtimes s_0 NG_2^0) \rtimes (s_1 NG_2^0 \rtimes s_1 s_0 NG_1^0)) \rtimes$$
$$((s_2 NG_2^0 \rtimes s_2 s_0 NG_1^0) \rtimes (s_2 s_1 NG_1^0 \rtimes s_2 s_1 s_0 NG_0^0)). \tag{2.1}$$

We are not going to spell out the rather complicated definition of a hypercrossed complex [6]. Instead, we give some examples.

Example 2.4. A 1-hypercrossed complex of Lie groups is the same thing as a Lie crossed module.

Definition 2.5. Let H and D be two Lie groups. We say that H is a crossed D-module if there is a Lie group morphism $\delta_1 : H \to D$ and a smooth action of D on H $(d, h) \mapsto {}^d h$ such that

$${}^{\delta_1(h)} h' = h h' h^{-1} \text{ (Peiffer condition)}$$

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[3]It is a normal subgroup of G_{n-1} too.

[4]The objects of the full subcategory of simplicial groups with Moore complex of length k are also called k-*hypergroupoids* of groups [12].

for $h, h' \in H$, and

$$\delta_1(^d h) = d\delta_1(h)d^{-1}$$

for $h \in H, d \in D$ hold true.

We will use the notation $H \overset{\delta_1}{\to} D$ or $H \to D$ for a crossed module.

Definition 2.6. A morphism between Lie crossed modules $H \overset{\delta_1}{\to} D$ and $H' \overset{\delta'_1}{\to} D'$ is a pair of Lie group morphisms $\lambda : H \to H'$ and $\kappa : D \to D'$ such that the diagram

$$
\begin{array}{ccc}
H & \overset{\delta_1}{\longrightarrow} & D \\
\lambda \downarrow & & \downarrow \kappa \\
H' & \overset{\delta'_1}{\longrightarrow} & D'
\end{array}
$$

commutes, and for any $h \in H$ and $d \in D$ we have the following identity

$$\lambda(^d h) = {}^{\kappa(d)}\lambda(h).$$

Starting from a Lie crossed module $H \to D$ we can build up the corresponding simplicial Lie group. Explicitly, cf. Proposition 2.3,

$$G_0 = D, \quad G_1 = (H \rtimes D), \quad G_2 = (H \rtimes (H \rtimes D)), \quad \text{etc.}$$

The construction can be interpreted as the internal nerve of the associated internal category in the category of Lie groups (a strict Lie 2-group).

Example 2.7. A Lie 2-hypercrossed complex is the same thing as a Lie 2-crossed module [7].

Definition 2.8. A Lie 2-crossed module is a complex of Lie groups

$$H \overset{\delta_2}{\to} D \overset{\delta_1}{\to} K \tag{2.2}$$

together with smooth left actions by automorphisms of K on H and D (and on K by conjugation), and the Peiffer pairing, which is an smooth equivariant map $\{ , \} : D \times D \to H$, i.e., $^k\{d_1, d_2\} = \{^k d_1, {}^k d_2\}$ such that:

i) (2.2) is a complex of K-modules, i.e., δ_2 and δ_1 are K-equivariant and $\delta_2\delta_1(h) = 1$ for $h \in H$,

ii) $d_1 d_2 d_1^{-1} = \delta_2\{d_1, d_2\} {}^{\delta_1(d_1)}d_2$, for $d_1, d_2 \in D$,

iii) $h_1 h_2 h_1^{-1} h_2^{-1} = \{\delta_2 h_1, \delta_2 h_2\}$, for $h_1, h_2 \in H$,

iv) $\{d_1 d_2, d_3\} = \{d_1, d_2 d_3 d_2^{-1}\} {}^{\delta_1(d_1)}\{d_2, d_3\}$, for $d_1, d_2, d_3 \in D$,

v) $\{d_1, d_2 d_3\} = {}^{d_1 d_2 d_1^{-1}}\{d_1, d_3\}\{d_1, d_2\}$, for $d_1, d_2, d_3 \in D$,

vi) $\{\delta_2(h), d\}\{d, \delta_2(h)\} = h^{\delta_1(d)}(h^{-1})$, for $d \in D, h \in H$,

wherein the notation $^k d$ and $^k h$ for left actions of the element $k \in K$ on elements $d \in D$ and $h \in H$ has been used.

There is an obvious notion of a morphism of Lie 2-crossed modules.

Definition 2.9. A morphism between Lie 2-crossed modules $H \xrightarrow{\delta_2} D \xrightarrow{\delta_1} K$ and $H' \xrightarrow{\delta'_2} D' \xrightarrow{\delta'_1} K'$ is a triple of smooth group morphisms $H \to H'$, $D \to D'$ and $K \to K'$ making up, together with the maps δ_2, δ'_2, δ_1 and δ'_1, a commutative diagram

$$\begin{array}{ccccc}
H & \xrightarrow{\delta_2} & D & \xrightarrow{\delta_1} & K \\
\lambda \downarrow & & \mu \downarrow & & \downarrow \nu \\
H' & \xrightarrow{\delta'_2} & D' & \xrightarrow{\delta'_1} & K'
\end{array} \tag{2.3}$$

and being compatible with the actions of K on D and H and of K' on D' and H', respectively and with the respective Peiffer pairings.

The corresponding simplicial Lie group is given explicitly by, cf. Proposition 2.3,

$$G_0 = K, \quad G_1 = (D \rtimes K), \quad G_2 = (H \rtimes D) \rtimes (D \rtimes K)),$$
$$G_3 = (H \rtimes (H \rtimes D)) \rtimes ((H \rtimes D) \rtimes (D \rtimes K)), \quad \text{etc.}$$

This can be interpreted as an internal Duskin nerve [10].

Example 2.10. A Lie 3-hypercrossed complex is the same thing as a Lie 3-crossed module of [4].

We refer the interested reader to [6] for a thorough discussion of hypercrossed complexes of groups and their relation to simplicial groups.

At each level n, there is an lexicographically ordered set $S(n)$ of 2^n sets, which defines the compositions of the degeneracy maps appearing in the decomposition of G_n. Explicitly for $S(n)$ we have:

$$\{\emptyset < \{0\} < \{1\} < \{1,0\} < \{2\} < \{2,0\} < \{2,1\} < \{2,1,0\} < \ldots < \{n-1,\ldots,1\} < \{n-1,\ldots,0\}\}.$$

The important role in the theory of hypercrossed complexes is played by the actions $G_0 \times NG_n \to NG_n$ defined by

$$g_0 \times g_n \mapsto {}^{g_0}g_n : (s_{n-1} \ldots s_0 g_0) g_n (s_{n-1} \ldots s_0 g_0)^{-1}$$

and the so called *Peiffer pairings*. In order to define these, we will use the multi-indices like α and β from $\bigcup_n S(n)$ to write s_α for products of degeneracy maps

$$s_0, \ s_1, \ s_1 s_0, \ s_2, \ s_2 s_0, \ s_2 s_1, \ s_2 s_1 s_0, \ldots$$

In particular, for $g \in NG_{n-\sharp\alpha}$ we have $s_\alpha g \in G_n$. For each n consider the set $P(n)$ of pairs (α, β) such that $\emptyset < \alpha < \beta$ and $\alpha \cap \beta = \emptyset$, where $\alpha \cap \beta$ is the set of indices belonging to both α and β.

Definition 2.11. The Peiffer pairing (or lifting) $F_{\alpha,\beta}(g,h) \in NG_n$ for $g \in NG_{n-\sharp\alpha}$, $h \in NG_{n-\sharp\beta}$ and $(\alpha,\beta) \in P(n)$ is defined by

$$F_{\alpha,\beta}(g,h) = p_n(s_\alpha(g)s_\beta(h)s_\alpha(g)^{-1}s_\beta(h)^{-1}),$$

where p_n is the projection to NG_n. For the projector p_n, we have $p_n = p_n^1 \ldots p_n^n$ with $p_n^i(g) = gs_{i-1}\partial_i g^{-1}$.

For us, the relevant Peiffer pairings at each level n will be those defined for pairs $(\alpha,\beta) \in P(n)$ such that $\alpha \cup \beta = \{0,\ldots n\}$. We shall denote the set of such pairs $\bar{P}(n)$.

Remark 2.12. For a simplicial Lie algebra $\mathfrak{g}$, we have the corresponding Moore complex $N\mathfrak{g}$ of Lie algebras, which carries a structure of a hypercrossed complex of Lie algebras, cf. [3]. All the definitions and statements of this section have, of course, their infinitesimal counterparts. Since these are obvious, we shall not formulate them explicitly.

As shown by Quillen [18] there is an adjunction between simplicial Lie algebras and dg-Lie algebras. The part of the adjunction going from simplicial Lie algebras to dg-Lie algebras acts on the underlying simplicial vector spaces as the Moore complex functor N.

3. Simplicial principal bundles

Let G be a simplicial Lie group and X a simplicial manifold. In this paper we use the name *principal G-bundle* for a *twisted Cartesian product*. Therefore, we start with defining twisting functions. Again, we will denote by ∂_i and s_i the corresponding face and degeneracy maps. We follow [16].[5]

Definition 3.1. For a smooth function $\tau : X_n \to G_{n-1}$ to be a twisting, the following conditions should be fulfilled:

$$\partial_0\tau(x)\tau(\partial_0 x) = \tau(\partial_1 x),$$
$$\partial_i\tau(x) = \tau(\partial_{i+1}x) \quad \text{for} \quad i > 0,$$
$$s_i\tau(x) = \tau(s_{i+1}x) \quad \text{for} \quad i \geq 0,$$
$$\tau(s_0 x) = e_n \quad \text{for} \quad x \in X_n.$$

Definition 3.2. Let τ be a twisting function. A twisted Cartesian product $P(\tau) = G \times_\tau X$ (alternatively a principal G-bundle, or simply G-bundle, $P \to X$) is the simplicial manifold with simplices

$$P(\tau)_n = G_n \times X_n$$

and with the following face and degeneracy maps

$$\partial_i(g,x) = (\partial_i g, \partial_i x) \quad \text{for} \quad i > 0,$$

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[5]Again, passing from sets to manifolds is straightforward.

$$\partial_0(g, x) = (\partial_0 g.\tau(x), \partial_0 x),$$
$$s_i(g, x) = (s_i g, s_i x) \quad \text{for} \quad i \geq 0.$$

The principal (left) G-action

$$G_n \times P(\tau)_n \to P(\tau)_n, \quad g'_n \times (g_n, x_n) \mapsto (g'_n g_n, x_n)$$

and the projection

$$\pi_n : P_n \to X_n, \quad (g_n, x_n) \mapsto x_n$$

are smooth simplicial maps.

Equivalence of two G-bundles $P(\tau)$ and $P(\tau')$ over the same X is described in terms of twisting as follows.

Definition 3.3. We call two twistings τ' and τ equivalent if there exists a smooth map $\psi : X \to G$ such that

$$\partial_0 \psi(x).\tau'(x) = \tau(x).\psi(\partial_0 x),$$
$$\partial_i \psi(x) = \psi(\partial_i x) \quad \text{if} \quad i > 0,$$
$$s_i \psi(x) = \psi(s_i x) \quad \text{if} \quad i \geq 0.$$

In particular a twisting or the corresponding G-bundle $P(\tau)$ is trivial iff

$$\tau(x) = \partial_0 \psi(x)^{-1}.\psi(\partial_0 x),$$

with ψ as above.

As with ordinary bundles, simplicial principal bundles can be pulled back and their structure groups can be changed using simplicial Lie group morphisms. Twistings transform under these operations in an obvious way.

There is a canonical construction of the classifying space $\overline{W}G$ and of the universal G-bundle WG.

Definition 3.4. The classifying space $\overline{W}G$ is defined as follows. $\overline{W}G_0$ has one element $*$ and $\overline{W}G_n = G_{n-1} \times G_{n-2} \times \ldots \times G_0$ for $n > 0$. Face and degeneracy maps are

$$s_0(*) = e_0, \qquad \partial_i(g_0) = * \quad \text{for} \quad i = 0 \text{ or } 1$$

and

$$\partial_0(g_n, \ldots g_0) = (g_{n-1}, \ldots, g_0),$$
$$\partial_{i+1}(g_n, \ldots, g_0) = (\partial_i g_n, \ldots, \partial_1 g_{n-i+1}, \partial_0 g_{n-i}.g_{n-i-1}, g_{n-i-2}, \ldots, g_0),$$
$$s_0(g_{n-1}, \ldots, g_0) = (e_n, g_{n-1}, \ldots, g_0),$$
$$s_{i+1}(g_{n-1}, \ldots, g_0) = (s_i g_{n-1}, \ldots, s_0 g_{n-i}, e_{n-i}, g_{n-i-1}, \ldots, g_0),$$

for $n > 0$. With the choice of a twisting given by

$$\tau(g_{n-1}, \ldots, g_0) = g_{n-1}$$

we have the universal G-principal bundle

$$WG = G \times_\tau \overline{W}G.$$

We have a relation between twistings and simplicial maps $X \to \overline{W}G$ given by the following proposition.

Proposition 3.5. *The map $f_\tau : X \to \overline{W}G$ given by*

$$x \mapsto * \quad for \quad x \in X_0$$

and

$$x \mapsto (\tau(x), \tau(\partial_0 x), \dots, \tau(\partial_0^i x), \dots, \tau(\partial_0^{n-1} x)) \quad for \quad x \in X_n, \, n > 0$$

is a smooth simplicial map.

Vice versa, a smooth simplicial map $f : X \to \overline{W}_G$, given by

$$x \mapsto * \quad for \quad x \in X_0$$

and

$$x \mapsto (g_{n-1}^{(n)}(x), \dots, g_0^{(n)}(x)) \quad for \quad x \in X_n, \, n > 0$$

defines a twisting by

$$\tau_f(x) = g_{n-1}^{(n)}(x) \quad for \quad x \in X_n, \, n > 0.$$

We have $\tau_{f_\tau} = \tau$ and $f_{\tau_f} = f$.

The role of the universal bundle is the following.

Theorem 3.6. *The principal G-bundle $G \times_\tau X$ corresponding to the twisting τ is obtained from the universal bundle WG as a pullback under the simplicial map f_τ.*

4. L_∞-ALGEBROIDS AS 1-JETS OF SIMPLICIAL SETS

This section is completely based on [21], to which we also refer for the proofs. We keep, maybe with an occasional exception, the notation and terminology used there. Let SSM denotes the category with objects being surjective submersions between supermanifolds and morphisms commutative squares. Any surjective submersion $M \to N$ gives a simplicial supermanifold X, the nerve of the the groupoid $M \times_N M \rightrightarrows N$. Further, let SSM_1 denote the full subcategory of SSM with objects $\mathbb{R}^{0|1} \times N \overset{\mathrm{pr}_2}{\rightrightarrows} N$, where N is running through all supermanifolds. Let $SM_{[1]}$ be the category of supermanifolds with a right action of the supersemigroup $\mathbf{Hom}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$. Put in other words, $SM_{[1]}$ is the category of differential non-negatively graded supermanifolds. We have the following lemma

Lemma 4.1. *The category $\widehat{SSM}_1$ of presheaves on SSM_1 and the category $\widehat{SM}_{[1]}$ of presheaves on $SM_{[1]}$ are equivalent.*

Remark 4.2. The above lemma follows from the useful observation

$$\mathrm{Hom}(\mathbb{R}^{0|1} \times N \to N, \mathbb{R}^{0|1} \times X \to X) \simeq \mathrm{Hom}(N, X) \times \mathbf{Hom}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(N).$$

Which just says that the object $\mathbb{R}^{0|1} \times X \in \mathrm{SSM}_1$ corresponds to the object $X \times \mathbf{Hom}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ in $\mathrm{SM}_{[1]}$.

Definition 4.3. Let F be a presheaf on SSM. Its restriction to SSM_1 is an object in $\widehat{\mathrm{SSM}_1}$. The corresponding object F_1 in $\widehat{\mathrm{SM}_{[1]}}$ is called the *1-jet* of F.

Remark 4.4. The representable 1-jets are of particular interest, since they are represented by differential non-negatively graded supermanifolds [6]. Hence, they can provide us with interesting examples of those. If the $\mathbb{Z}_2$ is given by the parity of the $\mathbb{Z}$-degree, which will be always the case in our examples, then we have a differential non-negatively graded manifold. Let us recall, that a finite-dimensional, positively graded differential manifold is the same thing as an L_∞-algebra. If it is only a non-negatively graded one then it could be, for good reasons explained in [21], referred to as an L_∞-algebroid, cf. also [23] for a formal definition.

Particular examples of presheaves on SSM come from simplicial supermanifolds. If K is a simplicial supermanifold and X the nerve of the groupoid defined by the surjective submersion $M \to N$, the the corresponding sheaf $F^K \in \widehat{\mathrm{SSM}}$ is defined by

$$F^K(M \to N) = \mathrm{Hom}(X, K),$$

i.e. it associates with the surjective submersion $M \to N$ the set of all simplicial maps $X \to K$.

In [21], also the following sufficient condition for the 1-jet F_1^K of F^K to be representable is given.

Theorem 4.5. *Let K be a simplicial supermanifold fulfilling the Kan conditions, which is moreover m-truncated for some $m \in \mathbb{N}$. Then the 1-jet F_1^K is representable.*[7]

Another construction described in [21] is the so called 1-approximation of a presheaf $F \in \widehat{\mathrm{SSM}}$. The restriction of F to SSM_1 admits a right adjoint, the induction.

Definition 4.6. The presheaf $\mathrm{app}_1 F \in \widehat{\mathrm{SSM}}$ is defined by successively applying the restriction and induction functors to $F \in \widehat{\mathrm{SSM}}$.

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[6]Differential graded supermanifolds, i.e. Q-supermanifolds provide a natural framework for the Batalin-Vilkovisky formalism [2].

[7]We refer to the proof as well for explanation of the Kan conditions and m-truncatedness to the Appendix of [21]. These notions were, for simplicial manifolds, first introduced in [14]. If G is a simplicial Lie group then G, $\overline{W}G$ and WG fulfill the Kan conditions[19].

Proposition 4.7. *If the jet 1-jet F_1 is represented by the differential non-negatively graded supermanifold X_F then the sheaf $\mathrm{app}_1 F \in \widehat{\mathrm{SSM}}$ is given by*

$$\mathrm{app}_1 F(M \to N) = \{\text{morphisms of dg manifolds } T[1](M \to N) \to X_F\}$$
$$= \{\text{morphisms of dg algebras } C^\infty(X_F) \to \Omega(M \to N)\},$$

where $T[1](M \to N)$ is the shifted fibrewise tangent bundle of M and $\Omega(M \to N) = C^\infty(T[1](M \to N))$ are the fibrewise differential forms on M.

If X_F is positively graded then it can be identified with an L_∞-algebra L_F and we have

$$\mathrm{app}_1 F(M \to N) = \{\mathrm{Maurer-Cartan\ elements\ of\ } L_F \otimes \Omega(M \to N)\}.$$

Example 4.8. Consider the presheaf in $\widehat{\mathrm{SSM}}$ represented by $Y \to X$. Its 1-jet is 1-representable by $T[1](Y \to X)$ (equipped with the canonical differential), the shifted fibrewise tangent bundle. This is just a fibrewise version of the following well known fact $\mathrm{Hom}(R^{0|1} \times N \to N, Y \to *) = \mathbf{Hom}(R^{0|1}, Y)(N) \cong \mathrm{Hom}(N, T[1]Y)$, i.e. that "maps from $\mathbb{R}^{0|1}$ to M are the same things as 1-forms on M".

Example 4.9. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $M \to N$ a surjective submersion. A G-descent data on $M \to N$, i.e a descent of a trivial G-bundle on M to a G-bundle on N, is a map $g : M \times_N M \to G$ satisfying $g(x,x) = e$ and $g(x,y)g(y,z) = g(x,z)$ for $(x,y,z) \in M \times_N M \times_N M$. The G-descent is the same thing as a groupoid morphism from $M \times_N M \to G$. Let us consider the presheaf

$$F(M \to N) = \{G - \text{descent data on } M \to N\},$$

which is in the above notation $F^{\mathcal{N}G}$, with $\mathcal{N}G$ the nerve of G.

By definition, we have for the 1-jet $F_1^{\mathcal{N}G}$

$$F(\mathbb{R}^{0|1} \times N \to N) = \{G - \text{descent data on } \mathbb{R}^{0|1} \times N \to N\}.$$

Such a G-descent data is a map $g : \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \to G^N$ satisfying the above descent (1-cocycle) condition and is equivalent to a map $\bar{g} : \mathbb{R}^{0|1} \to G^N$, such that $\bar{g}(0) = e$. The relation between maps g and $\bar{g}$ is [8]

$$g(\theta_0, \theta_1) = \bar{g}(\theta_0)^{-1}\bar{g}(\theta_1),$$
$$\bar{g}(\theta) = g(0, \theta).$$

One way to see what the dg manifold representing the 1-jet looks like is the following. Instead of imposing the condition $\bar{g}(0) = e$, we can consider arbitrary functions $\bar{g}(\theta)$ modulo left multiplications with the constant ones. So what we have is the shifted tangent bundle $T[1]G$ equipped with the canonical differential induced from the de Rham differential on G modulo the left G-action. This observation immediately leads to the dg algebra of functions on $\mathfrak{g}[1] \cong \mathbf{Hom}(\mathbb{R}^{0|1}, G)/G$ – the wedge algebra of left invariant

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[8]Notice that $\bar{g}$ is just a trivialization of the 1-cocycle g over $\mathbb{R}^{0|1}$.

forms on G with the de Rham differential, which is just the the Chevalley-Eilenberg complex of $\mathfrak{g}$.

Equivalently, with an obvious abuse of notation, which we will commit also in the rest of the paper, we note that we can write

$$\bar{g}(\theta) = e - a\theta,$$

with $a \in \mathfrak{g}^N[1]$. Hence, the 1-jet F_1 is represented by the shifted Lie algebra $\mathfrak{g}[1]$. The differential is computed from

$$g(\theta_0, \theta_1) = 1 + a(\theta_0 - \theta_1) + \frac{1}{2}[a, a]\theta_0\theta_1$$

by computing

$$-(da)\theta_1 = (\delta_\epsilon \bar{g}(\theta_1) - e) = \frac{d}{d\epsilon}(g(\theta_0 + \epsilon, \theta_1 + \epsilon) - e) \mid_{\epsilon = \theta_0 = 0} = -\frac{1}{2}[a, a]\theta_1.$$

Hence, the differential is

$$da = \frac{1}{2}[a, a].$$

Finally, the functor $\mathrm{app}_1 F$ associates to a surjective submersion $M \to N$ the set of flat fibrewise connections.[9]

Example 4.10. Let $H \overset{\delta_1}{\to} D$ be a crossed module of Lie groups with the induced crossed module of Lie algebras

$$\mathfrak{h} \overset{\delta_1}{\to} \mathfrak{d}$$

and $M \to N$ a surjective submersion. An $H \to D$-descent data on $M \to N$, is an $(H \to D)$-valued 1-cocycle on on the groupoid $Y = M \times_N M$. Such 1-cocycles describe bundle gerbes, similarly as transition functions describe principal bundles [1]. More explicitly, we have a pair of maps (h, d), $d : Y_1 \to D$ and $h : Y_2 \to H$, such that

$$d(y_1)d(y_2) = \delta_1(h(y_1, y_2))d(y_1 \circ y_2), \quad \text{for} \quad (y_1, y_2) \in Y_2,$$

$$h(y_1, y_2)h(y_1 \circ y_2, y_3) = {}^{d(y_1)}h(y_2, y_3)h(y_1, y_2 \circ y_3) \quad \text{for} \quad (y_1, y_2, y_3) \in Y_3,$$

and

$$d(e_x) = e \quad \text{and} \quad h(e_{s(y)}, y) = h(y, e_{t(y)}) = e.$$

Let us consider the presheaf

$$F(M \to N) = \{(H \to D) - \text{descent data on } M \to N\},$$

which is in the above notation $F^{\mathcal{N}(H \to D)}$, with $\mathcal{N}(H \to D)$ the Duskin nerve of $H \to D$.

By definition, we have for the 1-jet $F_1^{\mathcal{N}(H \to D)}$

$$F(\mathbb{R}^{0|1} \times N \to N) = \{(H \to D) - \text{descent data on } \mathbb{R}^{0|1} \times N \to N\}.$$

---

[9]Vice versa, if G and all the fibres are 1-connected flat fibrewise connections give us G-descents.

Such an $(H \to D)$-descent data is a pair of maps (h, d), $d : \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \to D^N$ and $h : \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \to H^N$ satisfying the 1-cocycle condition, and is equivalent to a pair of maps $(\bar{h}, \bar{d})$, $\bar{d} : \mathbb{R}^{0|1} \to D^N$ and $\bar{h} : \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \to H^N$ such that

$$\bar{d}(0) = e, \quad \bar{h}(\theta, \theta) = \bar{h}(0, \theta) = e.$$

The relation between pairs of maps (d, h) and $(\bar{d}, \bar{h})$ is[10]

$$d(\theta_0, \theta_1) = \bar{d}(\theta_0)^{-1} \delta_1(\bar{h}(\theta_0, \theta_1)) \bar{d}(\theta_1),$$

$$\bar{d}(\theta_0) h(\theta_0, \theta_1, \theta_2) = \bar{h}(\theta_0, \theta_1) \bar{h}(\theta_1, \theta_2)) \bar{h}(\theta_0, \theta_2))^{-1}$$

and

$$\bar{d}(\theta) = d(0, \theta),$$

$$\bar{h}(\theta_0, \theta_1) = h(0, \theta_0, \theta_1).$$

Obviously, we can write

$$\bar{d}(\theta) = e - a\theta,$$

with $a \in \mathfrak{d}[1]$ and

$$\bar{h}(\theta_0, \theta_1) = e + b\theta_0\theta_1,$$

with $b \in \mathfrak{h}[2]$. Hence, the 1-jet F_1 is represented by the graded vector space $\mathfrak{d}[1] \oplus \mathfrak{h}[2]$.

The differential is computed in a complete analogy with Example 4.9 using expressions

$$d(\theta_0, \theta_1) = e + a(\theta_0 - \theta_1) + (\frac{1}{2}[a, a] + \delta_1 b)\theta_0\theta_1,$$

$$h(\theta_0, \theta_1, \theta_2) = e + b(\theta_0\theta_1 + \theta_1\theta_2 - \theta_0\theta_2) - {}^a b \, \theta_0\theta_1\theta_2.$$

The resulting differential is:

$$da = \frac{1}{2}[a, a] + \delta_1 b,$$

$$db = {}^a b.$$

Since we have a positively graded dg manifold, we can describe it as an L_∞-algebra. It is actually a DGLA with generators only in lowest two degrees, i.e a strict Lie 2-algebra. The nonzero components are $L_0 = \mathfrak{d}$ and $L_{-1} = \mathfrak{h}$. The differential is $\delta_1 : \mathfrak{h} \to \mathfrak{d}$. The bracket on $\mathfrak{d}$ is given by its own Lie bracket, and the bracket between $\mathfrak{d}$ and $\mathfrak{h}$ is given by the by the action of $\mathfrak{d}$ on $\mathfrak{h}$. Let us note that Lie 2-algebras are one to one to crossed modules of Lie groups, cf. [5].

Finally, the functor $\mathrm{app}_1 F$ associates to a surjective submersion $M \to N$ the set of $(\mathfrak{h} \to \mathfrak{d})$-valued flat fibrewise connections.

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[10]Again, the pair $(\bar{h}, \bar{d})$ is just a trivialization of the 1-cocycle (h, d).

5. L_∞-ALGEBRA OF $\overline{W}G$

In this section we generalize the Examples 4.9 and 4.10 to the case of a G-descent, where G is a simplicial Lie group with Moore complex of length k. The associated simplicial Lie algebra will be denoted by $\mathfrak{g}$. Examples 4.9 and 4.10 correspond to $k = 1$ and $k = 2$ respectively. Let $M \to N$ be a surjective submersion. We define a G-descent data on $M \to N$ as a G-valued twisting on the nerve of the groupoid $N \times_M N$. We recall, cf. Definition 3.1, that for $\tau : X_n \to G_{n-1}$ to be a twisting, the following conditions should be fulfilled:

$$\partial_0 \tau(x) \tau(\partial_0 x) = \tau(\partial_1 x),$$

$$\partial_i \tau(x) = \tau(\partial_{i+1} x) \quad \text{for} \quad i > 0,$$

$$s_i \tau(x) = \tau(s_{i+1} x) \quad \text{for} \quad i \geq 0,$$

$$\tau(s_0 x) = e_n \quad \text{for} \quad x \in X_n.$$

Let us consider the presheaf

$$F(M \to N) = \{G - \text{descent data on } M \to N\},$$

which is in the notation of the previous section $F^{\overline{W}G}$, i.e. the sheaf associating with the surjective submersion $N \to M$ the set of all simplicial maps from the nerve of the groupoid $N \times_M N$ to the classifying space $\overline{W}G$.

By definition, we have for the 1-jet $F_1^{\overline{W}G}$

$$F(\mathbb{R}^{0|1} \times N \to N) = \{G - \text{descent data on } \mathbb{R}^{0|1} \times N \to N\}.$$

Such an G-descent data is described by a twisting [11] $\tau : (\mathbb{R}^{0|1})^n \to G_{n-1}^N$ and is equivalent to a function $\psi : (\mathbb{R}^{0|1})^n \to G_n^N$ such that

$$\partial_i \psi(\theta_0, \ldots \theta_n) = \psi(\theta_0, \ldots, \hat{\theta}_i, \ldots \theta_n) \quad \text{if} \quad i > 0,$$

$$s_i \psi(\theta_0, \ldots \theta_n) = \psi(\theta_0, \ldots, \theta_i, \theta_i \ldots \theta_n) \quad \text{if} \quad i \geq 0.$$

We have the following relation between τ and ψ [12]

$$\tau(\theta_0, \ldots \theta_n) = \partial_0 \psi(\theta_0, \ldots, \theta_n)^{-1} \psi(\theta_1, \ldots, \theta_n),$$

$$\psi(\theta_0, \ldots, \theta_n) = \tau(0, \theta_0, \ldots, \theta_n).$$

From the definition of ψ it follows that

$$\psi(0, \theta_1, \ldots, \theta_n) = \tau(0, 0, \theta_1, \ldots, \theta_n) = \tau(s_0(0, \theta_1, \ldots, \theta_n)) = e_n.$$

Therefore, we write

$$\psi(\theta_0, \ldots, \theta_n) = 1 - a(\theta_1, \ldots \theta_n)\theta_0,$$

---

[11]From now on we will omit the annoying N and assume it everywhere implicitly.

[12]As before, ψ is a trivialization of τ, cf. Definition 3.3.

with $a(\theta_1, \ldots \theta_n) \in \oplus_{i=0}^{n} \binom{n}{i} \mathfrak{g}_{\mathfrak{n}}[i+1]$. The function a fulfils the following identities

$$\partial_i a(\theta_1, \ldots \theta_n) = a(\theta_1, \ldots, \hat{\theta}_i, \ldots \theta_n) \quad \text{if} \quad i > 0, \tag{5.1}$$

$$\partial_0 a(0, \theta_1, \ldots \theta_n) = a(\theta_1, \ldots \theta_n) \tag{5.2}$$

$$s_i a(\theta_1, \ldots \theta_n) = a(\theta_1, \ldots, \theta_i, \theta_i, \ldots, \theta_n) \quad \text{if} \quad i > 0, \tag{5.3}$$

$$s_0 a(\theta_1, \ldots, \theta_n) = a(0, \theta_1, \ldots, \theta_n). \tag{5.4}$$

In the above list, the only possibly not completely obvious one is the the ∂_0 equation (5.2). However, this one follows from the s_0 equation (5.4) by an application of ∂_0. From (5.1) we immediately see that

$$a^n \in N\mathfrak{g}_n[n+1],$$

for the top component a^n of $a(\theta_1, \ldots \theta_n) = a^n \theta_1 \ldots \theta_n + \ldots$.

To proceed further, it will be more convenient change the Grassmann coordinates by $\bar{\theta}_0 = \theta_1$ and $\bar{\theta}_i = \theta_{i+1} - \theta_i$ for $i > 1$. In terms of $\bar{\theta}$s, we get the following lemma for the decomposition of $a(\bar{\theta}_0 \ldots \bar{\theta}_{n-1}) \in \oplus_{i=0}^{n} \binom{n}{i} \mathfrak{g}_n[i+1]$ in terms of the shifted Moore complex $N\mathfrak{g}_k[k+1] \oplus \ldots \oplus N\mathfrak{g}_0[1]$

Lemma 5.1. *For $n \leq k$*

$$a(\bar{\theta}_0, \ldots, \bar{\theta}_{n-1}) = \sum_{\alpha \in S(n)} s_\alpha a^{n - \sharp \alpha} \bar{\theta}^{S(n) \setminus \alpha},$$

where $\bar{\theta}^\beta := \bar{\theta}_{i_1} \ldots \bar{\theta}_{i_l}$ for $\beta = \{i_n, \ldots, i_1\} \in S(n)$.

Proof. Straightforward computation using the fact that with the new Grassmann variables $\bar{\theta}$ we have nice simplicial relations $s_i a(\bar{\theta}_0, \ldots \bar{\theta}_{n-1}) = a(\bar{\theta}_0, \ldots, \bar{\theta}_{i-1}, 0, \bar{\theta}_i, \ldots, \bar{\theta}_{n-1})$ and $s_{i-1} \partial_i a(\bar{\theta}_0, \ldots, \bar{\theta}_{n-1}) = a(\bar{\theta}_0, \ldots, \bar{\theta}_{n-1})|_{\bar{\theta}_{i-1}=0} \square$.

We see that, for $n \leq k$, the only independent component of $a \in \oplus_{i=0}^{n} \binom{n}{i} \mathfrak{g}_{\mathfrak{n}}[i+1]$ is the top one $a^n \in \mathfrak{g}_{\mathfrak{n}}[n+1]$. Hence, the the 1-jet F_1 in this case is represented by $N\mathfrak{g}_k[k+1] \oplus \ldots \oplus N\mathfrak{g}_0[1]$ as a graded manifold.

The differential can be obtained in analogy with the Examples 4.9 and 4.10. We write

$$\tau(\theta_0, \ldots, \theta_n) = e + \partial_0 a(\theta_1, \ldots \theta_n)\theta_0 - a(\theta_2, \ldots, \theta_n)\theta_1 + \frac{1}{2}[\partial_0 a(\theta_1, \ldots \theta_n), a(\theta_2, \ldots, \theta_n)]\theta_0\theta_1$$

$$= e + \partial_0 a(\theta_1, \ldots, \theta_n)\theta_0 - a(\theta_2, \ldots, \theta_n)\theta_1 + \frac{1}{2}[\partial_0 a(0, \theta_2 \ldots \theta_n), a(\theta_2, \ldots, \theta_n)]\theta_0\theta_1$$

$$= e + \partial_0 a(\theta_1, \ldots, \theta_n)\theta_0 - a(\theta_2, \ldots, \theta_n)\theta_1 + \frac{1}{2}[a(\theta_2, \ldots \theta_n), a(\theta_2, \ldots, \theta_n)]\theta_0\theta_1.$$

Now, using the above expression for τ,

$$-(da(\theta_2, \ldots, \theta_n))\theta_1 = \delta_\epsilon(\psi(\theta_1, \ldots, \theta_n) - e) = \frac{d}{d\epsilon}(\tau(\theta_0 + \epsilon, \ldots, \theta_n + \epsilon) - e)\mid_{\epsilon=\theta_0=0}$$

$$= \frac{d}{d\theta_1}\partial_0 a(\theta_1, \ldots, \theta_n)\theta_1 + \sum_{i=2}^{n} \frac{d}{d\theta_i}a(\theta_2, \ldots, \theta_n)\theta_1 - \frac{1}{2}[a(\theta_2, \ldots, \theta_n), a(\theta_2, \ldots, \theta_n)]\theta_1.$$

Hence, the differential is

$$da(\theta_1, \ldots, \theta_n) = -\frac{d}{d\theta_0}\partial_0 a(\theta_0, \ldots, \theta_n) - \sum_{i=1}^{n} \frac{d}{d\theta_i} a(\theta_1, \ldots, \theta_n)$$

$$+ \frac{1}{2}[a(\theta_1, \ldots, \theta_n), a(\theta_1, \ldots, \theta_n)]. \tag{5.5}$$

Now, we proceed in extracting the action da^n of differential d on the top component a^n. For this note: The first term gives $-\partial_0 a^{n+1}$ and the second doesn't contribute to da^n at all . What remains is to determine the top component of the commutator $[a(\theta_1, \ldots, \theta_n), a(\theta_1, \ldots, \theta_n)]$. This leads to the following proposition.

Proposition 5.2. *Let G be a simplicial group with the simplicial Lie algebra $\mathfrak{g}$. Assume that its Moore complex NG is of length k. Then the 1-jet F_1 of the simplicial manifold $\overline{W}G$ is representable by the dg manifold $\oplus_{n=0}^{k} N\mathfrak{g}_n[n+1]$. The differential da^n on $a^n \in N\mathfrak{g}_n[n+1]$ is described in terms of the face map ∂_0, commutator of $\mathfrak{g}_0$, action of $N\mathfrak{g}_0$ on $N\mathfrak{g}_n$ and Peiffer pairings $f_{\alpha,\beta}$ with $(\alpha, \beta) \in \bar{P}(n)$ as follows:*

For $n = 0$

$$da^0 = -\partial_0 a^1 + \frac{1}{2}[a^0, a^0],$$

for $n > 0$

$$da^n = -\partial_0 a^{n+1} + {}^{a^0}a^n + \sum_{(\alpha,\beta)\in\bar{P}(n)} \pm f_{\alpha,\beta}(a^{n-\sharp\alpha}, a^{n-\sharp\beta}),$$

where the sign is given by the product of parity of $(n - \sharp\alpha)(n - \sharp\beta + 1)$ and the parity of the shuffle defined by the pair $(S(n) \setminus \alpha, S(n) \setminus \beta)$.

Proof. The 0th component is clear. What is left is to justify the form of the second and third term in the above expression for da^n, $n > 0$. However, this is easily done using the above Lemma 5.1. We just have to be careful about the degrees and signs. We have

$$\frac{1}{2}[a(\bar{\theta}_0, \ldots, \bar{\theta}_{n-1}), a(\bar{\theta}_0, \ldots, \bar{\theta}_{n-1})]^n = [s_{n_1} \ldots s_0 a^0, a^n] + \sum_{(\alpha,\beta)\in\bar{P}(n)} \pm[s_\alpha a^{n-\sharp\alpha}, s_\beta a^{n-\sharp\beta}],$$

with the sign given as the product of parities of $(n - \sharp\alpha)(n - \sharp\beta + 1)$ and of the shuffle defined by the pair $(S(n) \setminus \alpha, S(n) \setminus \beta)$. Of course $\sharp\alpha + \sharp\beta = n$.

The first term is just ${}^{a^0}a^n$, i.e. describing the action of $N\mathfrak{g}_0$ on $N\mathfrak{g}_0$ shifted by 1 in degree. Further, note that, by construction, the face ∂_i and degeneracy maps s_i commute with the differential d. In particular, it follows that da^n must be in $\in \bigcap_{i>0} \ker \partial_i[n+2]$. Moreover, since $\partial_i\partial_0 = \partial_0\partial_{i+1}$, we also have $\partial_0 a^{n+1} \in \bigcap_{i>0} \ker \partial_i[n+2]$. Therefore, we conclude that the sum over pairs $(\alpha, \beta) \in \bar{P}(n)$ in the above equation is also in $\in \bigcap_{i>0} \ker \partial_i[n+2]$ and as such can be written, trivially inserting the projection $p_n : \mathfrak{g}_n \to N\mathfrak{g}_n$, as $\sum_{(\alpha,\beta)\in\bar{P}(n)} \pm p_n[s_\alpha a^{n-\sharp\alpha}, s_\beta a^{n-\sharp\beta}]$. $\square$

It is now straightforward to describe the L_∞-algebra corresponding to the above dg manifold explicitly. What we have is a k-term DGLA $L = \oplus_{n=0}^{k} L_{-n}$ with components in degrees $0, -1, \ldots -k$, given by $L_{-n} = N\mathfrak{g}_n$. The differentials $d_n : N\mathfrak{g}_n \to N\mathfrak{g}_{n+1}$ are given by the restrictions $d_n = \partial_0|_{N\mathfrak{g}_n}$ of the zeroth face maps, i.e by the differentials δ_n of the Moore complex $N\mathfrak{g}$, i.e, for $x_n \in N\mathfrak{g}_n$

$$d_n x_n = \delta_n x_n. \tag{5.6}$$

The only nonzero brackets are the binary brackets. The nonzero binary brackets are determined by the following prescription:

The bracket $N\mathfrak{g}_0 \times N\mathfrak{g}_0 \to N\mathfrak{g}_0$ is just the Lie bracket on $N\mathfrak{g}_0$, i.e for $x_0 \in N\mathfrak{g}_0$ and $y_0 \in N\mathfrak{g}_0$

$$[x_0, y_0]. \tag{5.7}$$

The brackets $N\mathfrak{g}_0 \times N\mathfrak{g}_n \to N\mathfrak{g}_n$: $(x, y) \mapsto [x_0, x_n] = -[x_n, x_0]$ are given by the action of $N\mathfrak{g}_0$ on $N\mathfrak{g}_n$

$$[x_0, x_n] = -[x_n, x_0] = {}^{x_0}x_n. \tag{5.8}$$

The bracket $N\mathfrak{g}_{n_1} \times N\mathfrak{g}_{n_2} \to N\mathfrak{g}_n$ with $n = n_1 + n_2$, for n_1 and n_2 nonzero, is described as follows: For $x_{n_1} \in N\mathfrak{g}_{n_1}$ and $x_{n_2} \in N\mathfrak{g}_{n_2}$

$$[x_{n_1}, x_{n_2}] = \sum_{(\alpha,\beta)\in\bar{P}(n_1,n_2)} \pm f_{\alpha,\beta}(x_{n_1}, x_{n_2}) + (-1)^{(n_1+1)(n_2+1)} \sum_{(\alpha,\beta)\in\bar{P}(n_2,n_1)} \pm f_{\alpha,\beta}(x_{n_2}, x_{n_1})$$
$$\tag{5.9}$$

The $\pm$ sign is given by the product of parity of $n_1(n_2 + 1)$ and the parity of the shuffle defined by the pair $(\alpha, \beta) \in \bar{P}(n_1, n_2)$. Here $\bar{P}(n_1, n_2) \subset \bar{P}(n)$ denotes the subset of $P(n)$ consisting of those pairs $(\alpha, \beta) \in \bar{P}(n)$, for which $n - \sharp\alpha = n_1, n - \sharp\beta = n_2$.

Let us now consider an arbitrary simplicial Lie algebra $\mathfrak{g}$ with Moore complex of length k. Associated to $\mathfrak{g}$ we have the (unique) simplicial group G integrating it, such that all its components are simply connected. Therefore, starting with $\mathfrak{g}$, we can consider the functor $F_1^{\overline{W}G}$. Correspondingly, we have the following theorem:

Proposition 5.3. *Let $\mathfrak{g}$ be a simplicial Lie group with Moore complex $N\mathfrak{g}$ of length k. Then $N\mathfrak{g}$ or becomes a DGLA. The differential and the binary brackets are explicitly given by formulas (5.6-5.9). This DGLA structure on $N\mathfrak{g}$ is the same one as described by Quillen's construction in Proposition 4.4 of* [18].

We finish with some remarks:

Remark 5.4. Obviously, one can reformulate the above theorem in terms of a k-hypercrossed complex of Lie algebras $\underline{\mathfrak{g}}$. Such a k-hypercrossed complex $\underline{\mathfrak{g}}$ has a structure of a k-term DGLA described by (5.6-5.9).

Remark 5.5. As noted above, crossed modules and Lie 2-algebras are one to one. From the above theorem, we see that for $n > 2$ only a part of the full hypercrossed complex structure enters the description of the DGLA L. For instance, already for $n = 3$, only the symmetric part of the Peiffer pairing appears. Nevertheless, the simplicial (Kan) manifold $\overline{W}G$ can be interpreted as an integration of the DGLA $N\mathfrak{g}$.

On the other hand, any L_∞-algebra L, in particular any DGLA, can be integrated to a (Kan) simplicial manifold $\int L$ [14]. So one might try to compare the integration $\int N\mathfrak{g}$ with the Kan simplicial manifold $\overline{W}G$, or the corresponding 1-jets (differentiations). Here we restrict ourselves only to two related (obvious) remarks.

First, there is the following observation: Let M be a simplicial manifold. Assume that its corresponding 1-jet functor F_1^M is representable by an L_∞ algebra L, with Chevalley-Eilenberg complex $C(L)$. Also, let $\Omega(\Theta_N)$ be DGA of (normal) forms on the simplicial supermanifold Θ_N, the nerve associated to the surjective submersion $\mathbb{R}^{0|1} \times N \to N$.[13] Then, for the 1-jet corresponding to $\int L$, we have $\mathrm{Hom}(\Theta_N, \int L) = \{$morphisms of dg algebras $C(L) \to \Omega(\Theta_N)\}$. This has to be compared to 1-jet corresponding to M, i.e. to the set $\mathrm{Hom}(\Theta_N, M) = \{$morphisms of dg algebras $C(L) \to \Omega(\mathbb{R}^{0|1} \times N \to N)\}$.

Second, in [14], simplicial homotopy groups $\pi_n^{\mathrm{spl}} \int L$ of $\int L$ have been shown to be finite dimensional diffeological groups. Lie algebra of a diffeological group is defined as the Lie algebra of its universal cover, which is a Lie group. It is a result of [14] that the Lie algebra of $\pi_n^{\mathrm{spl}} \int L$ is canonically isomorphic to $H_{n-1}(L)$. Specified to the case of our interest: the Lie algebra of $\pi_n^{\mathrm{spl}} \int N\mathfrak{g}$ is canonically isomorphic to $H_{n-1}(N\mathfrak{g})$. If we now consider $\overline{W}G$ just as a simplicial set then for its simplicial homotopy groups we have $\pi_n \overline{W}G = \pi_{n-1}G = H_{n-1}(NG)$. Given the Lie structure of G, $H_{n-1}(NG)$ and hence $\pi_n \overline{W}G$ can be considered as diffeological groups. In this sense we can talk about the Lie algebra of $\pi_n \overline{W}G$, which is again $H_{n-1}(N\mathfrak{g})$.

Remark 5.6. The functor $\mathrm{app}_1 F^{\overline{W}G}$ associates to a surjective submersion $M \to N$ the set of L-valued flat fibrewise connections. To obtain also non-flat connections one may use the Weil algebra of L, similarly as in [20]. This, as well as applications to higher gauge theory, will be described elsewhere.

Remark 5.7. Regarding applications to higher gauge theory, in the forthcoming work we plan to extend the results presented here to the case of simplicial groupoids. In particular, we hope to describe a proper generalization of the Atiyah groupoid to the simplicial case and then obtain the Atiyah L_∞-algebroid as a 1-jet of a properly defined simplicial classifying space. This would lead us directly to the notion of an L-valued connection on a simplicial principal G-bundle (or on the corresponding hypercrossed complex bundle

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[13]See, e.g., [9] for the definition of forms on simplicial sets.

gerbe). That this should be possible can be seen from the description of connections and curvings of crossed module bundle gerbes in [22], where a categorification of the Atiyah algebroid is presented.

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